Compact anisotropic stars with membrane - a new class of exact solutions to the Einstein field equations

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Abstract

I report on a new class of solutions to the classical field equations of general relativity with zero cosmological constant. The solutions describe a spherically symmetric, compact self gravitating object with a smooth interior matter-distribution, residing in a static electro-vacuum space time.

An outstanding feature of the new solutions is a sharp, non-continuous boundary of the matter-distribution, which is accompanied by a membrane consisting of pure tangential pressure (surface tension/stress). The interior matter state generally has a locally anisotropic pressure.

A procedure to generate various new solutions is given. Some particular cases are derived explicitly and discussed briefly. To single out the physically most promising solutions, a selection principle based on the holographic principle is formulated.

One solution of particular interest emerges. The so called holographic solution, short holostar, is characterized by the property, that the "stress-energy content" of the holostar's boundary membrane is equal to its gravitating mass. The holostar's real membrane has identical properties to the - fictitious - membrane attributed to a black hole by the membrane paradigm. This feature guarantees, that the holostar's dynamical action on the exterior space-time is practically identical to that of a black hole with the same mass. The holostar's interior matter state can be interpreted as a collection of tightly packed radial strings attached to the spherical boundary membrane. The properties of the holographic solution are discussed in detail in three parallel papers. The solution provides a singularity free alternative to black holes and the standard cosmological models.

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1 Introduction

Black holes are among the simplest objects of theoretical physics, fully describable by just a few parameters. According to our present knowledge, a black hole is the final, most compact state of any sufficiently large gravitationally bound object.

Because of their subtle links to phenomena of quantum origin, such as Hawking radiation and entropy, many researchers view black holes as important theoretical probes to explore some aspects of the "new" physics beyond the Standard Model. The study of black holes plays a major role not only in classical relativity, but also in loop quantum gravity (LQG) and string theory. Despite much progress several open questions of black hole physics remain, mostly relating to their inner structure.

Our limited understanding of self gravitating objects can be traced to the fundamental problem, that so far there is no satisfactory theoretical foundation from which the "matter side" of gravity can be constructed by first principles.

A - possibly subjective - survey of recent and not so recent research results seems to point to the direction, that the concept of boundary areas might play a fundamental role in our future understanding of gravity, on the classical as well as on the quantum level.

Bekenstein was probably the first to realize the importance of boundary areas in classical general relativity. His conjecture, that the entropy of a black hole should be proportional to its event horizon [3] was spectacularly confirmed by Hawking [14], who demonstrated that black holes have a definite temperature, by which the constant of proportionality could be fixed.

In the recent past insights of a similar kind were achieved on the quantum level. The discovery of area-quantization in loop quantum gravity [26] can be viewed as one of the most important achievements in this field. In essence, area-quantization in non-perturbative background-free LQG is derived from the assumption, that the physical world is fundamentally relational (no background- or pre-geometry), and should most naturally be described in terms of quantities which are invariant with respect to the artifacts of the particular description that we impose. The relevance of boundary areas has also become apparent in the string approach. Certain solutions of string theory suggest, that the dynamical degrees of freedom in an internally consistent string theory are proportional to the boundary of a space-time region. This conjecture has become known to a larger audience under the popular name of "holographic principle" [33, 32].

Whereas the "geometric" side of gravity can be considered to be fairly well understood, the incorporation of "matter" into gravity is still a matter of debate. The question with respect to the fundamental (matter) degrees of freedom in a self-consistent theory of gravity has not been answered. The black hole solutions of classical general relativity give paradoxical answers. Whereas the fundamental degrees of freedom of a black hole are associated with its horizon, the "mass" of the space-time is associated with the central singularity. In string theory the fundamental degrees of freedom are known, at least in principle. But we have not (yet) been able to locate a single string (or brane) in the physical

world. The loop approach so far has focussed on the geometry, which is quantized without reference to matter. Although there is some evidence that the fundamental results of LQG, such as area and volume quantization, will remain valid if matter is added to the theory [25, 30], it is not yet clear how this is to be done.

Loop quantum gravity gives beautiful answers to what the outcome of the measurement of an area will be, but it leaves physicists quite in the dark "what area" actually should be measured. Area operators referring to different (coordinate) surfaces generally do not commute [27]. Intuitively we are accustomed to define the location of a surface via matter. Diffeomorphism invariance however suggest, that any meaningful area must refer to matter. Unfortunately the classical vacuum black hole solutions of general relativity do not contain matter (or fields) that could be associated with a surface.

The problem of "what we should measure" is a fundamental one. Many researchers view the event horizon of a black hole as the surface, to which some answers of quantum gravity, such as the quantization of the area, should be applied.[1] However, this appears not altogether consistent and happens to be in direct contradiction to one of the most successful fundamental principles of physics, relationism. Relationism demands that space-time events, or more generally space-time regions, must be defined by matter.[2] The famous "hole argument" of classical general relativity, given by Einstein in 1912 [10] demonstrates quite clearly, that the measurement of a distance, an area or volume in empty space doesn't make sense, i.e. doesn't yield an unambiguous result. If relationism is a fundamental property of the physical world, and if areas play an important role in gravity, any physically meaningful area should be defined by matter and should be localizable through matter.

What form could such matter-states take? It has been known for quite some time that in a spherically symmetric context different space-times regions, such as an interior de-Sitter core and an exterior electro-vac region, can be smoothly matched through a so called transition region. The not necessarily thin transition layer generally has a substantial surface pressure/tension, whose integrated "energy-content" can become comparable to or even exceed the asymptotic gravitational mass of the object (for a very general discussion of such space-times, encompassing rotating sources, see [6] and the references therein). In [13] formula for the surface energy-density and surface pressures for an infinitesimally thin transition region were given, however no explicit solutions were derived. One particular solution, the so called "gravastar", that has been been put forward recently by [20], has received some considerable interest. The gravastar consists of a localizable thin, spherical shell of matter separating an interior de Sitter condensate phase from an exterior Schwarzschild vacuum. The entropy of the gravastar, however, is calculated to be orders of magnitude lower than the Hawking entropy of a black hole of the same gravitating mass. Whereas the Hawking entropy scales with area ($S \propto r^2 \propto M^2$), the entropy of the gravastar

[1] see for example [5]

[2] Alternatively: by localizable properties of the fields, which may be just the other side of the same coin.

scales with its square-root ($S \propto r \propto M$). It will remain to be seen, whether such a substantial deviation from the established theoretical framework of black hole physics will find acceptance.

The concept of Hawking entropy is theoretically well founded and has been derived through several independent ways. The essential result, the entropy-area law, has its foundations in the fundamental "Four Laws of Black Hole Mechanics"[2, 3, 4]. The entropy-area law has been confirmed by calculations in string theory [16, 19, 31], as well as in the loop approach [1]. Therefore in this work the position is taken, that the thermodynamic properties of a compact, self gravitating object are - at least approximately - correctly described by the Hawking-entropy and -temperature, irrespective of its interior structure.

However, identifying the enormous entropy of a black hole with its event horizon, a vacuum region locally indistinguishable from the surrounding vacuum space-time, whose location can only be determined by (global) knowledge of the whole space-time's future, seems to be incompatible with our well established conceptions about causality[3] and the fundamental nature of entropy. In fact, the statistical entropy of a macroscopic system is amongst the most rigorously defined physical concepts, exactly calculable if the microscopic constituents and their interactions are known.

This leaves us with several open questions. What is the true origin of the Hawking-entropy of a black hole? Where do the microstates of a black hole reside and how do they interact? Where has the constituent matter gone? Does it and how does it participate to the entropy and to the asymptotic gravitational mass and/or field? What is the origin of (gravitational) mass? Are the sources of the gravitational field localizable, and if yes, to what extent (to a point, string or surface)? What role do boundary areas play? And finally: What are the fundamental "matter" states, from which the matter side of gravity can be constructed from first principles?

Recent research in string theory, as well as in loop quantum gravity, have provided some very important insights. Classical general relativity so far has not been very successful in providing satisfying and consistent answers. Classical general relativity has been plagued with various (apparent) paradoxes, most of which can be traced to presence of an event horizon in vacuum and its "one-way membrane-property".

In this paper exact solutions to the field equations are derived, which might help to resolve some of the paradoxes. The properties of the solutions might also be helpful to advance the understanding of the "matter side" of gravity.

[3] The event horizon "moves" a-causally in anticipation of the matter that will eventually pass it in the future.

2 Field Equations for a Spherically Symmetric System with Locally Anisotropic Pressure

The approach taken in this paper is to derive the components of the stress-energy tensor for the most general, isolated, spherically symmetric self gravitating object. Neither will I assume that the pressure is locally isotropic, nor that the matter-fields (i.e. mass-density ρ and the three principle pressures P_i) are continuous. However, the metric is assumed to be continuous everywhere.

With respect to notation I closely follow the presentation in [35], using the standard spherical metric in the (+ - - -) sign-convention, with geometric units $c = G = 1$:

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2 \tag{1}$$

The most general form of the stress-energy tensor for a spherically symmetric (static) system is constructed out of three independent matter/pressure fields, which, due to spherical symmetry, do not depend on θ and φ, have the same magnitude in $\partial\theta$- and $\partial\varphi$-direction, and, due to the requirement of staticness, are independent from a suitably chosen - time coordinate t:

$$T_\mu^\nu(r) = \begin{pmatrix} \rho(r) & & & \\ & -P_r(r) & & \\ & & -P_\theta(r) & \\ & & & -P_\theta(r) \end{pmatrix} \tag{2}$$

Due to spherical symmetry the trace of the stress-energy tensor, T, only depends on the radial coordinate r. The r-dependence will be dropped in the formula, whenever appropriate:

$$T = T_\mu^\mu = \rho - P_r - 2P_\theta \tag{3}$$

With T the field equations read as follows:

$$R_{\mu\nu} = -8\pi\left(T_{\mu\nu} - \frac{T}{2}\right) \tag{4}$$

The components of the Ricci-tensor can be calculated from the metric coefficients and their first and second derivatives. Due to spherical symmetry the field equations reduce to three equations for the three matter-fields:[4]

$$R_{tt} = R_{00} = -\frac{B''}{2A} + \frac{B'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{rA} = -4\pi B(\rho + P_r + 2P_\theta) \tag{5}$$

$$R_{rr} = R_{11} = \frac{B''}{2B} - \frac{B'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rA} = -4\pi A(\rho + P_r - 2P_\theta) \tag{6}$$

[4] see for example [35, p. 300] or [12, p. 128]

$$R_{\theta\theta} = R_{22} = -1 - \frac{r}{2A}\left(\frac{A'}{A} - \frac{B'}{B}\right) + \frac{1}{A} = -4\pi r^2(\rho - P_r) \tag{7}$$

3 A general procedure for the generation of spherically symmetric solutions

Combining equations (5) and (6) gives the following expression:

$$\frac{A'}{A} + \frac{B'}{B} = \frac{(AB)'}{AB} = (\ln(AB))' = 8\pi Ar(\rho + P_r) \tag{8}$$

Thus, in a spherically symmetric space-time, any extended region with $AB = const$ gives rise to an equation of state $\rho + P_r = 0$ and vice versa.

Equation (8) allows us to eliminate B'/B in equation (7), giving the following differential relation between A and ρ:

$$(\frac{r}{A})' = \frac{1}{A} - \frac{rA'}{A^2} = 1 - 8\pi r^2 \rho \tag{9}$$

It is quite a remarkable feature, due to spherical symmetry, that the radial metric coefficient, A, only depends on the mass-density, $\rho(r)$, even in the case of anisotropic pressure.

Integration of (9) gives the well known expression for the radial metric coefficient of a spherically symmetric gravitationally bound object:

$$A(r) = \frac{1}{1 - \frac{2M(r)}{r}} \tag{10}$$

with

$$M(r) = M_0 + \int_0^r 4\pi r^2 \rho(r) dr \tag{11}$$

A point mass M_0 at the origin has been included as an integration constant.

The time coefficient of the metric B can be calculated from A and the two matter fields ρ and P_r by means of integrating equation (8). The integration is usually performed starting at $r = \infty$ and setting $B(\infty) = 1$. From a fundamental viewpoint it is more natural to take the event horizon (when it exists) as the starting point for any integration of the field equations.

The event horizon with its finite proper area and fixed topology is an almost ideal "anchorpoint" for the space-time geometry, at least for the spherically and axially symmetric vacuum solutions. If there is vacuum outside the event horizon, the starting values for the integration of equations (9) and (8) at the event horizon ($r = r_+$) are unambiguously known: $r_+/A(r_+) = 0$ and $lnAB(r_+) = 0$.

In the case of a space-time with no event horizon, the boundary of the matter-distribution, r_h, should be taken as starting point of the integration of equations (9) and (8). We then still have $lnAB(r_h) = 0$, if the exterior vacuum space-time is asymptotically flat. If there is no event horizon the integration constant

$r_h/A(r_h) = r_0$ can in principle take on any positive value. The more compact a self-gravitating object becomes, the smaller r_0 will be. The most compact self-gravitating object without event-horizon is expected to have $r_0 \approx r_{Pl}$.

The general procedure for determining the metric of a compact, spherically symmetric object is:

(i) Determine A by the following definite integral:

$$\frac{r}{A} = r_0 + \int_{r_h}^{r} 1 - 8\pi r^2 \rho dr \tag{12}$$

(ii) Determine B by:

$$\ln(AB) = \ln AB(r_h) + \int_{r_h}^{r} 8\pi Ar(\rho + P_r)dr \tag{13}$$

Although it is preferable to integrate the field equations starting out from the boundary of the compact object, in this paper more often than not the equation for A will be integrated in the "usual" way, i.e. starting out from $r = 0$, using a point mass M_0 at the origin as an integration constant. This will make it easier to compare the parameters of the solution, such as the value of its point mass at the origin, with the well known Schwarzschild solution.

A and B can be derived by integrals involving only ρ and P_r without any explicit reference to the tangential pressure P_θ. This seems to imply that the metric only depends on ρ and P_r. A closer analysis shows that the dependencies are more subtle. The integration constant of physical importance is M_0 (or rather r_0). The global scale factor for B has no physical significance. However, any specific choice of M_0 (or r_0) fixes the tangential pressure and vice versa, so - in fact - the metric coefficients A and B depend unambiguously on all three matter-pressure fields.

The tangential pressure P_θ can be calculated from the metric, either from equations (5) and (6) or by the following equation:

$$8\pi P_\theta = \frac{1}{AB}(\frac{B''}{2} + \frac{B'}{r}) - 2\pi(\rho + P_r)(\frac{rB'}{B} + 2) \tag{14}$$

In the general case, however, the tangential pressure is most conveniently determined by the continuity equation, which in the case of locally anisotropic pressure ($P_r \neq P_\theta$) is given by:

$$\frac{1}{A}({P_r}' + \frac{2P_r}{r}) + \frac{B'}{2AB}(P_r + \rho) - \frac{2P_\theta}{rA} = 0 \tag{15}$$

As long as $1/A$ has no zeros (i.e. a space-time without event horizon) the tangential pressure P_θ can be derived from the continuity equation via:

$$P_\theta = P_r + \frac{r{P_r}'}{2} + \frac{rB'}{B}(P_r + \rho) \tag{16}$$

If the metric is known, all matter fields can be derived from the metric coefficients via equations (9, 8, 14).

The matter-density ρ and the radial pressure P_r depend only on the first derivatives of the metric, whereas P_θ depends also on the second derivatives. This has important consequences on the structure of the solutions. The matter-fields of the new solutions generally contain step-discontinuities (θ-distributions) and non-regular δ-distributions. But only the tangential pressure may have a δ-distribution, otherwise the metric could not be continuous. Continuity of the metric is essential for the mathematical structure of general relativity, whereas the "gravitational field", i.e. the metric derivatives, may contain finite jumps without severe mathematical consequences.

In the following ρ and P_r are assumed to have at most finite jumps. This guarantees, that the event horizon (alternatively: the boundary of the matter-distribution) can be chosen without any ambiguity as starting point for the integration of the metric coefficients A and B.

The general procedure to generate solutions for the spherically symmetric case, which will be followed in this paper, is the following:

- Guess a radial dependence for the mass-density $\rho(r)$. The mass-density can be non-continuous, but should not contain any δ-distributions.

- Calculate the radial metric coefficient $A(r)$ by integrating equation (9) with a particular choice of point mass at the origin M_0. Alternatively a particular choice of $r_0 = r_h/A(r_h)$ at the boundary of the matter-distribution is used.[5]

- Determine the radial pressure $P_r(r)$ from $\rho(r)$ by assuming a particular equation of state. This allows one to calculate the time coefficient of the metric, $B(r)$ via integration of equation (8). If the matter-distribution is bounded and the exterior space-time asymptotically flat, the integration constant for B is given by $\ln AB(r_h) = 0$.

- Finally determine the tangential pressure $P_\theta(r)$ from the first and second derivatives of the metric (14) or by the continuity equation (16).

A vast number of solutions can be constructed by this procedure. Any well behaved function for the energy-density ρ and for the equation of state $P_r(\rho)$ will lead to a solution. Various analytical or numerical solutions can be found in the literature.[6]

In this paper I will briefly discuss some solutions, which I found interesting from a physical or mathematical perspective. The choice is subjective. I present the main arguments that have led to the particular choice of solution, the so called "holographic solution" of section 4.5, which is discussed in greater detail in [23, 22, 24]. A thorough discussion of some of other new solutions which might be of interest must be left to forthcoming papers.

[5] If the boundary of the matter-distribution, r_h, coincides with the position of the event horizon, i.e. $r_h = r_+$, the integration constant r_0 must be zero.

[6] Algorithms similar or equivalent to the above described procedure can be found in the recent literature. These rather general approaches including anisotropic pressure states appear to have been developed quite recently and apparently independently by different authors, see for example [6, 8, 9, 11, 13, 15, 18, 28].

4 Solutions with equation of state: $\rho + P_r = 0$

Solutions with the above equation of state can be regarded as a natural extension of the Schwarzschild and Reissner-Nordström solutions:

The Schwarzschild and Reissner-Nordström solutions satisfy the constraint $AB = 1$ throughout the whole space-time.From the (multiplicative) constraint on the metric, $AB = const$, the (additive) constraint on the fields, $\rho + P_r = 0$, follows via equation (8).

A number of physically interesting solutions appear to lie in this restricted class of solutions, or are approximated by this class. If we assume that the mass-energy density ρ is positive, the radial pressure will be negative for this type of solutions. This shouldn't be a problem: An equation of state with a negative pressure is well established in inflational cosmology. Recent results on the large scale distribution of matter in the universe indicate a "dark matter" component with overall negative pressure $p \simeq -0.8\rho$ (see for example [21, p. 244]), which - within the errors - comes close to the equation of state $\rho + P_r = 0$. Furthermore, an equation of state with negative pressure has been proposed in the not so recent past for the Lorentz invariant vacuum state, see for example [36]. $\rho + P_r = 0$ also is the equation of state for a cosmic string, with ∂r being the longitudinal direction of the string.

Imposing the constraint $\rho + P_r = 0$ on the solutions greatly simplifies the math. Equations (5), (6) and (7) are reduced to the following set of equations:

$$(rB)' = 1 - 8\pi r^2 \rho \tag{17}$$

and

$$8\pi P_\theta = \frac{B''}{2} + \frac{B'}{r} \tag{18}$$

By differentiating equation (17) or by using the continuity equation (16) the tangential pressure can be expressed solely in terms of the mass-density:

$$8\pi P_\theta = \frac{B''}{2} + \frac{B'}{r} = \frac{(rB)''}{2r} = -8\pi(\rho + \frac{r\rho'}{2}) \tag{19}$$

The system of three nonlinear equations in the two metric coefficients A and B and their first and second derivatives has been transformed into a linear, albeit inhomogeneous set of differential equations of second order in B.

Equation (17) has the following - well known - solution:

$$B(r) = 1 - \frac{2M(r)}{r} \tag{20}$$

I will now make a rather special ansatz for the matter-density $\rho(r)$:

$$\rho = \frac{c}{8\pi r^2}\theta(r - r_+) \tag{21}$$

I.e. the mass-density follows an inverse square-law up the horizon, r_+, where it drops to zero in one discontinuous step. c is a dimensionless constant, not to

be confused with the velocity of light. Note that in this section the boundary of the mass-distribution, which will be denoted by r_h, coincides with the radial coordinate position of the gravitational radius of the object, $r_+ = 2M$. This is not mandatory. In the following sections solutions with $r_h \neq r_+$ will be studied.

With hindsight, the ansatz in equation (21) can be "justified" by the following arguments:

- Avoidance of trapped surfaces

  We would like to avoid trapped surfaces in our solutions. On the other hand, we are interested in the most compact, bounded matter-distribution possible. The second condition requires that the exterior space-time is vacuum up to (or almost up to) the gravitational radius of the object, which in the case of $r_+ = r_h$ coincides with the event horizon. To fulfill the first condition, i.e. to avoid trapped surfaces, the interior matter must extend to the event horizon with $\rho \geq 1/(8\pi {r_+}^2)$ at the event horizon. This can be seen as follows:

  If there were vacuum at the "inner" side of the event horizon, the coefficients A and B of the exterior Schwarzschild metric and their derivatives would be continuous throughout the horizon. Due to the finite slope of $1/A$ at the horizon and $1/A(r_+) = 0$ the radial coordinate inevitably becomes time-like for $r < r_+$. A time-like coordinate cannot stand still for any material object, thus all radial distances must shrink. Vacuum "behind" the horizon implies that any concentric sphere within this vacuum region is a trapped surface. From the singularity theorems it follows that a singularity inevitably will form. If trapped surfaces and therefore singularities are to be avoided a sufficiently positive mass-density must be "placed" directly at the inner side of the horizon in such a way, that the jump in the mass density will "turn up" the slope of $1/A$ such, that $1/A$ does not undergo a sign change. It is easy to see from equation (9), that any mass density $\rho \geq 1/(8\pi {r_+}^2)$ will do.

- Scale invariant mass-density

  The gravitational structures in the universe appear to be scale-invariant, at least on scales large enough so that gravity has become the dominant mechanism for structure formation. Whenever possible, a scale-invariant mass-energy density should be preferred in the solutions.

  A mass-density following an inverse square law is scale invariant (at least in a spherically symmetric context). In units $G = c = 1$ mass (or energy) has dimensions of length. Thus from dimensional considerations a "natural" mass-energy density is expected to have dimensions of inverse length squared. Only such a mass-energy density is truly scale-invariant.[7] Any other functional form of the mass density requires at least one dimensional constant, which would introduce a specific length scale.

[7]This is not quite true: In units $c = \hbar = 1$ frequently used by particle physicists, mass has dimensions of inverse length. Therefore an energy density $\rho \propto 1/r^4$ can be considered as scale-invariant as well, however only on a microscopic, not on a macroscopic scale.

With ρ given by equation (21) the mass-function $M(r)$ can be determined by a simple integration:

$$M(r) = \{ \begin{array}{ll} M_0 + c\frac{r}{2} & r < r_+ \\ M_0 + c\frac{r_+}{2} & r \geq r_+ \end{array} \tag{22}$$

By another integration one arrives at the following expression for the time coefficient of the metric, B:

$$B(r) = \{ \begin{array}{ll} 1 - c - \frac{2M_0}{r} & r < r_+ \\ 1 - \frac{2M_0 + cr_+}{r} & r \geq r_+ \end{array} \tag{23}$$

This result can be expressed by means of observable quantities, such as the gravitating mass M of the black hole or - alternatively - its gravitational radius $r_+ = 2M$:

Outside of the source-region, i.e. for $r > r_+$ the solution must be identical to the well known Schwarzschild vacuum-solution, due to Birkhoff's theorem. By comparing the Schwarzschild solution with the above solution, one can relate the gravitating mass M of the Schwarzschild solution to the parameters of the solution given by equation (23):

$$M = M_0 + \frac{cr_+}{2} \tag{24}$$

The well known identity between gravitational radius and gravitating mass, $r_+ = 2M$, then gives the following relation between the "interior" parameter M_0 (point mass at the origin) and the measurable "exterior" parameter M (gravitating mass of the black hole) and the dimensionless scaling factor, c, for the mass-density:

$$M_0 = M(1 - c) \tag{25}$$

The case $c = 0$ is special. Here the point mass M_0 at the origin exactly equals the gravitating mass M of the black hole. The interior matter-fields are zero. This is what is expected, if the $c = 0$ solution corresponds to the known Schwarzschild vacuum solution.

In the following the metric will be expressed in terms of r_+ and c:

$$B(r) = \{ \begin{array}{ll} (1 - c)(1 - \frac{r_+}{r}) & r < r_+ \\ 1 - \frac{r_+}{r} & r \geq r_+ \end{array} \tag{26}$$

The tangential pressure can be derived from the metric. For calculational purposes it is convenient to combine the interior and exterior solutions with help of the Heavyside-step function (θ-distribution) into one equation:

$$B(r) = (1 - \frac{r_+}{r})(1 - c + c\theta(r - r_+)) \tag{27}$$

With equation (18) and by exploiting the well known properties of the δ- and θ-distributions, i.e. $\theta' = \delta$ and $B(r_+)\delta(r - r_+) = 0$, we arrive at the following result for the tangential pressure:

$$P_\theta = \frac{c}{16\pi r_+}\delta(r - r_+) = \frac{M - M_0}{32\pi M^2}\delta(r - r_+) \tag{28}$$

The reader can easily check by substituting $B(r)$, $A(r) = 1/B(r)$, $\rho(r)$, $P_r(r)$ and $P_\theta(r)$ into the field equations (5), (6) and (7), that the just derived solutions satisfy the original field-equations and the continuity equation exactly in a distributional sense.

The peculiar behavior of the tangential pressure is one of the outstanding features of the new solutions, e.g. $c \neq 0$. An inverse square law for the interior energy-density ρ leads to an interior tangential pressure identical zero, except for a δ-distribution at the horizon. The horizon, constituting the boundary of the matter-distribution, is localizable, either through its property as the dividing line between the matter-filled interior region and the exterior Schwarzschild vacuum region, or alternatively by the membrane's surface tension/pressure. The interior matter is subject to a negative, purely radial pressure, identical to the pressure of a classical string in the radial direction ($P_r = -\rho; P_\perp = 0$).

The active gravitational mass of the interior space-time is zero, as expected from the equation of state of a cosmic string. Thus it appears as if the "net energy content" of the interior matter-fields comes out zero. However, this does not take into account the "energy content" of the horizon and the point mass M_0 at the center.

The value of c determines the tangential pressure at the position of the horizon. The integration constant M_0 depends directly on c and on the gravitating mass M via equation (25). Therefore M_0 cannot be chosen independently from the tangential pressure. In fact, the tangential pressure at the space time's boundary and the point mass at the origin are intimately related. From a physical point of view it is the tangential pressure at the event horizon which determines M_0. The important physical question therefore is, what value the tangential pressure should have at the horizon (or alternatively, what value the point mass M_0 at the center should have.)

This question can only be answered in an affirmative way by comparing the properties of solutions with different c, i.e. different "horizon-pressure" (or different point-masses at the origin), and choosing the solution whose properties are a better approximation to the observation.

Yet theory might give some guidance in the selection of the physically most relevant solution. There are two solutions, $c = 0$ and $c = 1$, which "stand out".

The "historic" Schwarzschild solution ($c = 0$) has been intensively studied. Its "horizon pressure" is zero, giving the Schwarzschild-solution the advantage of an analytic metric, except for the the singular point at $r = 0$. The Schwarzschild solution has the severe disadvantage of a singularity at $r = 0$ (breakdown of predictability). But both phenomena are related: In a spherically symmetric space-time with event horizon and with an equation of state $\rho + P_r = 0$ a zero horizon pressure ($c = 0$) is necessarily accompanied by a large point-mass at the origin ($M_0 = M$).

For the $c = 1$ solution the event horizon consists of a real physical membrane with tangential pressure $P_\theta = 1/(16\pi r_+)$. Due to the non-zero "horizon

pressure" the metric is not analytic at the horizon, not even differentially continuous. However, there is no point-mass at the origin and thus no singularity, which makes this type of solution particularly attractive from a physical perspective.

Quite remarkably, the pressure of the membrane for the $c = 1$ case is exactly equal to the tangential pressure attributed to a black hole by the membrane paradigm [34]. According to the membrane paradigm for black holes all (exterior) properties of a (vacuum) black hole can be explained by a (fictitious) membrane situated at the event horizon. Therefore, viewed by an exterior observer, both situations, $c = 0$ and $c = 1$ are completely indistinguishable. From a physical point of view it appears very attractive to substitute the fictitious membrane with the real membrane of the $c = 1$ solution. This leaves all important results derived for black holes with respect to the exterior space-time unchanged, and at the same time introduces exactly what appears to be needed (a zero point mass at the origin) to get rid of the severe problems regarding the interior space-time of a classical black hole, which is characterized by singularities, trapped surfaces and causally-disconnected regions. However, we will see in section 4.5 that getting rid of space-time singularities is not quite as easy as it seems. It requires one other ingredient: The elimination of the event horizon.

4.1 The weighted superposition principle

The equation of state $\rho + P_r = 0$, which is equivalent to $AB = const$, allowed us to linearize the Einstein-equations in the spherically symmetric case.

The linearity of the equations allows us to generate new solutions by (weighted) superposition. If B_1 is a solution for the fields ρ_1, P_{r1} and $P_{\theta 1}$ and B_2 is a solution for the fields ρ_2, P_{r2} and $P_{\theta 2}$, then the weighted sum of the metric coefficients $c_1 B_1 + c_2 B_2$ will be a solution for the respective weighted sum of the fields $c_1 \rho_1 + c_2 \rho_2, c_1 P_{r1} + c_2 P_{r2}$ and $c_1 P_{\theta 1} + c_2 P_{\theta 2}$, if the "norm condition" $c_1 + c_2 = 1$ is met. It should be noted, that even though B_1 and B_2 are genuine solutions of the field equations, $c_1 B_1$ or $c_2 B_2$ generally are not, because the field equations are inhomogeneous in the absence of sources. However, for the "fundamental" mass-density $\rho = 1/(8\pi r^2)$ the equations are rendered homogeneous.

Before some more general solutions to the equation of state $\rho + P_r = 0$ are derived, I will briefly discuss three special cases.

4.2 $c = 0$: the Schwarzschild-solution

For $c = 0$ the mass-density and pressures are identical zero, except at the center which contains a point mass equal to the gravitating mass of the black hole. The metric is identical to the well known Schwarzschild vacuum solution. Therefore the Schwarzschild vacuum solution constitutes a special case within the broader class of solutions characterized by $AB = const$ or $\rho + P_r = 0$.

4.3 $c = 2$: the negative-mass solution

This solution has a negative point mass at the origin, exactly equal but opposite in sign to the gravitating mass, i.e $M_0 = -M$. The interior metric coefficients A and B are the absolute values of the respective metric coefficients of the classical Schwarzschild vacuum solution.[8] The radial distance coordinate r remains space-like and the time-coordinate t time-like throughout the whole space-time. However, there is an ambiguity at the event horizon r_+, as $B(r_+) = 0$.

The weak, strong, dominant and null energy conditions are satisfied at any space-time *point*, except at the origin and at the horizon. The weak energy condition is only violated at a single point, the origin. Disregarding this singular point one could view the $c = 2$ solution as physically acceptable. However, in this work I take the position, that it doesn't make sense to attribute meaning to any physical quantity evaluated at a space-time point. Any physically meaningful space-time region will have a boundary area greater than the minimum non-zero area eigenvalue of loop quantum gravity. A violation of a physical principle, such as the weak or positive energy condition at a space-time point isn't considered problematic, as long as the violation is confined to a region smaller than Planck size. Turning the argument around, I propose that any physically acceptable classical space-time should be subject to the following conditions:

- In a physically acceptable classical space-time any physical quantity must be evaluated with respect to a physically meaningful space-time region.[9]
- Each relevant physical quantity or physical condition, evaluated for any physically meaningful space-time region, must lie within the accepted range for this quantity.[10]

With the above proposition in mind it is instructive to evaluate some of the properties of the $c = 2$ solution. The improper integral of the mass-density, including the negative point-mass at the origin, over any concentric interior sphere with "radius" r is given by:

$$M(r) = -\frac{r_+}{2} + r$$

[8] The "negative mass" solution can be derived by the weighted superposition principle of section 4.1: Superposition of twice the singular metric solution ($M_0 = 0$), discussed in the next chapter, minus the Schwarzschild solution ($M_0 = M$) gives the "negative mass" solution. The weights, 2 and -1, add up to 1.

[9] Note that this condition implies, that it must always be possible to associate a physical quantity with a "physically meaningful" space-time region. In the classical context the association will be through the proper integral of the quantity over the region. Other types of association, such as the improper integral, are possible. In the context of quantum gravity the appropriate operators have to be used. Throughout this paper I consider a "meaningful" space-time region to be a region of roughly Planck-volume bounded by a surface of roughly Planck area.

[10] For example, in a classical context the weak energy condition must hold for any arbitrary region of the space-time larger than Planck-size.

We find that the mass-function, a measure for the gravitational mass of the region, is negative for any sphere with $r < r_+/2$. The positivity of the energy is violated in this region.

The weak energy condition should be evaluated by the improper integral:

$$\int (\rho - P_r) 4\pi r^2 dr \geq 0$$

We find that the integrated weak energy condition is violated for any interior space-time region including the origin, due to the negative point mass situated at $r = 0$.

Therefore with respect to the above proposition the $c = 2$ solution is not physically acceptable, although it satisfies the relevant physical conditions at all of its space-time points, except at a collection of space-time points of measure zero (here: the origin, the horizon).

An analysis of the geodesic motion of a test-particle gives further evidence, that the region $r < r_+/2$ is physically unacceptable.

For the $c = 2$ solution the time coefficient B of the metric is equal to the absolute value of the time coefficient of the classical Schwarzschild interior metric. Thus the effective potential given in equation (96) of the Appendix reads:

$$V_{eff} = |\frac{r - r_+}{r_i - r_+}| \frac{r_i}{r} \left(1 - \beta_i^2 (1 - \frac{r_i^2}{r^2}) \right) \tag{29}$$

r_i is the position of the interior turning point of the motion and β_i is the tangential velocity of the particle at the turning point of the motion, expressed as a fraction to the local velocity of light.

Independent of the constants of the motion, r_i and β_i, the potential energy attains its minimum value at the horizon. Particles close to the horizon will undergo radially bounded oscillations around the radial position of the horizon, r_+. For particles with an interior turning point r_i of the motion close to the horizon the oscillations are bounded, irrespective of β_i, i.e. even for photons. Therefore we can expect, that small (radial) density fluctuations in the vicinity of the horizon will not be able to destabilize the black hole.

Any particle whose interior turning point r_i of the motion is less than $r_+/2$, has - formally - a radial velocity at infinity, which is greater than zero, as can be seen by evaluating the expression $\beta_r^2(r) = 1 - V_{eff}(r) \geq 0$ at $r = \infty$:

$$r_i \leq \frac{r_+}{2 - \beta_i^2} \tag{30}$$

Therefore any particle emanating from $r < r_+/2$ can in principle escape to infinity, although in general it will have to make it over the angular momentum barrier situated in the exterior space-time.

Particles with high β_i, i.e. with high tangential velocity at the turning point of the motion, are "unbound" for radial coordinate values larger than $r_+/2$. For photons with $\beta_i^2 = 1$ the inequality of equation (30) becomes $r_i < r_+$. Therefore any zero-rest mass particle, wherever situated in the black hole's interior,

can tunnel through the angular momentum barrier and permanently escape. Most massless particles don't even need to tunnel. They just pass over the barrier: For zero rest-mass particles the angular momentum barrier is situated at $3r_+/2$. The effective potential has a local maximum at this point, which is given by $V_{eff}(3r_+/2) = 4/27(r_i/r_+)^2/(r_+/r_i - 1)$. The effective potential has been normalized to $V_{eff}(r_i) = 1$. Therefore whenever $V_{eff}(3r_+/2) \leq 1$, the photons need not tunnel through the angular momentum barrier, because the radial component of their velocity at the barrier is greater than zero. This is the case for all zero rest-mass particles with:

$$r_i \leq \frac{3}{2} \frac{(1+\sqrt{2})^{\frac{2}{3}} - 1}{(1-\sqrt{2})^{\frac{1}{3}}} r_+ \cong 0.894 r_+ \tag{31}$$

For highly relativistic non-zero rest-mass particles a similar relation holds. The black hole can be expected to loose much of its inner mass. On the other hand, the time to pass through the horizon (either way) is infinite, as viewed by an asymptotic exterior observer. Therefore classically this type of black hole might still be considered stable.

Quantum mechanically the position of the horizon should be subject to fluctuations on the order of the Planck length. The exterior time t to approach the horizon up to a Planck-length (starting out from the angular momentum barrier at $r = 3r_+/2$)[11] is roughly given by $t \approx r_+ \ln r_+/r_{Pl}$. For a black hole of the size of the sun ($r \approx 10^{38} r_{Pl}$) this time is less than one micro-second.

Therefore it can be expected, that the black hole will quickly "loose" all of its "inner" particles.

This result must be interpreted with some reserve. There are two main objections. First, loop quantum gravity, with its prediction of quantized geometry, might not turn out to be the true quantum theory of gravity. This position is particularly advocated by string theorists. In this case there might not be any Planck-scale fluctuations of the geometry near the horizon, so that particles cannot "slip out" under the event horizon with $t \approx r_+ \ln r_+/r_{Pl}$. Second, in a space-time with pressure particles generally don't move on geodesics, as the particles are subject to non-vanishing pressure forces in their respective local inertial frames.

The first objection is as yet undecided. With respect to the second objection one might argue in a very handwaving manner, that although the interior particles won't move on geodesics, it seems quite unlikely that an interior particle acquiring a non-zero velocity at for example $r_i \cong r_+/2$ will be slowed down enough by the pressure to prevent its escape: For sufficiently large black holes, both mass-density and pressure in the region $r_+/2 \leq r \leq r_+$ become arbitrarily low. Therefore one might expect that the pressure not very much reduces the likeliness of escape. A detailed analysis will prove, whether this handwaving argument is essentially correct.

[11] At and beyond the angular momentum barrier the time-dilation due to the gravitational field is negligible, compared to the time dilation at the horizon: $B(3r_+/2) = 1/3 \gg B(r_+ + r_{Pl}) \simeq r_{Pl}/r_+$.

4.4 $c = 1$: the "singular metric" solution

An interesting case, at least mathematically, is the solution with $c = 1$. It has the following, well behaved fields:

$$\rho = -P_r = \frac{1}{8\pi r^2}(1 - \theta(r - r_+)) \tag{32}$$

$$P_\theta = \frac{1}{16\pi r_+}\delta(r - r_+) \tag{33}$$

However, its metric is singular within the whole interior region:

$$B(r) = (1 - \frac{r_+}{r})\theta(r - r_+) \tag{34}$$

$B(r)$ is identical zero within the interior according to equation (34). $A(r)$ is given by the inverse of $B(r)$. Thus within the entire black hole's interior "time stands still" ($B = 0$) and all proper radial distances are infinite ($A = \infty$).

It can be objected, that the above solution is not a genuine solution to the field equations at all, but rather a limiting case, separating the two genuine solutions classes with a negative or a positive point mass M_0 at the center of symmetry respectively. On the other hand, if the matter fields are regarded as distributions, operating on the function space of (continuous) test functions spanned by the metric coefficients, the solution should be at least mathematically well defined.

Physically the above solution is not very satisfactory, due to the singular metric.[12] It is doubtful, whether the "singular metric solution" can be viewed as an approximate description of a real physical system.

4.5 The holographic solution

A modification of the above singular solution with similar properties, but a non-singular interior metric can be constructed as follows:

As before let us consider space-time to be divided into two regions, separated by a spherical boundary at radial coordinate position r_h. The interior region is filled with "matter", the exterior region is vacuum.

The mass density within the interior region is assumed to be equal to the "natural" mass density $\rho = 1/(8\pi r^2)$ of the "singular metric" solution:

[12] Note however, that some of the undesirable features of the solution might be mended. For example, the bad behavior of the interior metric can be removed if the signature of the metric is changed from Lorentzian to Euclidian at the horizon. For a metric which is completely Euclidian (interior and exterior), the field equations yield a non-singular solution for the interior metric coefficients A and B with the matter-fields given in equations (32, 33). The utterly "frozen" state of the interior matter, due to $B = 0$, might even be regarded as a desirable feature of the solution. Some researchers have postulated a "frozen state", a so called "Planck-solid", inside a black hole [17]. Other researchers have pointed out, that quantum gravity predicts extended space-time regions with $B = 0$ (see for example [7, p. 92]). Also note, that with "time standing still" and "radial distances being infinite" quantum mechanical vacuum fluctuations should be heavily suppressed, even if the metric is allowed to fluctuate. This could provide an snbr-explanation for the cosmological constant problem.

$$\rho(r) = \frac{1}{8\pi r^2}(1 - \theta(r - r_h)) \tag{35}$$

In contrast to the "singular metric" solution, which does not have a point mass at the origin, a (small) point mass M_0 is assumed at the center.[13] Integration of equation (17) gives the following interior solution:

$$B_i(r) = \frac{-2M_0}{r} \tag{36}$$

The exterior solution outside the matter-filled region must be equal to the Schwarzschild solution, with a gravitating mass $M = r_+/2$:

$$B_e(r) = 1 - \frac{r_+}{r} \tag{37}$$

The metric must be continuous at the boundary r_h. The interior and exterior metric coefficient B shouldn't have a sign-change, i.e. should be positive everywhere. If the point mass at the origin M_0 is negative one can match the interior solution with $B > 0$ to the exterior solution, so that $r_h > r_+$, i.e. the boundary lies outside of the object's gravitational radius r_+:

$$r_h = r_+ + (-2M_0) \tag{38}$$

It is not possible to determine M_0 in the context of classical general relativity. With the two constants of the theory, c and G, no universal constant with the dimensions of mass (or length) can be constructed. Therefore, in the purely classical context it would be reasonable to assume that M_0 depends linearly on M. This puts r_h at a constant proportion with respect to r_+, leading to a "scale" invariant black hole, which "looks" the same on any scale. A natural choice would be $M = M_0$, which leads to $r_h = r_+$, i.e. the classical Schwarzschild vacuum solution.

If M_0 is postulated to be independent of M, the only "natural" classical choice - without a universal constant of the dimension of mass - is $M_0 = 0$, which leads to the "singular metric solution". Quantum theory, however, provides a third universal constant of nature, $\hbar$. From the three constants $\hbar$, c and G a universal constant with the dimension of mass can be constructed. Therefore, if M_0 is postulated to be constant and non-zero, the only natural choice that remains is the Planck mass. For the following I will set

$$-2M_0 = r_0$$

and assume, that r_0 is a positive quantity corresponding to the Planck-distance, except for a numerical factor of order unity. The tangential pressure then is given by:

[13] This is always possible, despite the formula of equation (25): $M_0 = (1 - c)M$. This formula was derived under the assumption that the boundary of the object coincides with the event horizon. Here we assume, that r_h may lie at a different position as the gravitational radius r_+, presumably outside of r_+.

$$P_\theta = \frac{1}{16\pi r_h}\delta(r - r_h) \tag{39}$$

with

$$r_h = r_+ + r_0 = 2M + r_0$$

The holographic solution satisfies all of the field equations, including the continuity equation. The exterior metric in the matter-free vacuum region is exactly that of a black hole with a gravitating mass $M = r_+/2$.

In the following pages the term compactar (= compact star) will be used to distinguish between the two classes of self gravitating systems: (i) solutions with event horizon (black holes) and (ii) the new type solutions, which have their gravitational radius marginally inside the matter-distribution, i.e. $r_h - r_+ = r_0 \approx r_{Pl}$.

The holographic solution appears to be the most interesting case of a compactar from a physical perspective. Its properties are discussed briefly in this paper, and somewhat more in detail in [23, 22, 24]. It turned out, that the holographic solution can explain many of the phenomena attributed to black holes, such as the Hawking entropy and temperature, and at the same time has much in common with the universe as we see it today. Its obvious string-character suggests that black holes, and possibly even the universe itself might be constructed hierarchically out of its basic building blocks, which appear to be strings and membranes (and particles). It therefore seemed appropriate to give the solution a name by which it can be referenced without much overhead. Henceforth I will refer to the holographic solution as "holostar" (= holographic star).

4.6 Black hole solutions with equation of state $P_r = -\rho$ and a power law in ρ

The emphasis in this section is on black hole solutions, i.e. solutions for which the radial coordinate position of the boundary r_h and the gravitational radius r_+ coincide. The case $r_h \neq r_+$ will be covered in the following section.

The mass-density ρ is expressed as a power law in the radial coordinate r:

$$\rho = -P_r = \frac{c}{8\pi r^2}(\frac{r_+}{r})^n(1 - \theta(r - r_h)) \tag{40}$$

In the following calculations the argument $(r - r_h)$ in the δ- and θ-distributions will be omitted.

The radial metric coefficient $1/A$ can be derived from the mass-density (40) by integrating equation (9)

$$B(r) = \frac{1}{A(r)} = \big(1 - \frac{r_+}{r}(1 - \frac{c}{1-n}) - \frac{c}{1-n}(\frac{r_+}{r})^n\big)(1 - \theta) + (1 - \frac{r_+}{r})\theta \tag{41}$$

This solution is valid for all powers n, with the exception $n = 1$, which is discussed separately. The solutions have been constructed such that B is continuous at the horizon. The integration was started out from the horizon, setting the integration constant $r_0 = 0$.

In general the solutions behave like a point-mass at the origin with M_0 given by:

$$M_0 = \frac{r_+}{2}(1 - \frac{c}{1-n}) = M(1 - \frac{c}{1-n}) \tag{42}$$

If M_0 is to be zero or close to zero for arbitrary M, this requires $c \simeq 1 - n$.

The tangential pressure can be derived from the metric coefficient B and its derivatives via equation (18):

$$P_\theta = \frac{cn}{16\pi r^2}(\frac{r_+}{r})^n(1-\theta) + \frac{c}{16\pi r_+}\delta \tag{43}$$

For $n = 1$ the metric and the tangential pressure are given by:

$$B(r) = \frac{1}{A(r)} = (1 - \frac{r_+}{r}(1 - c\ln\frac{r_+}{r}))(1-\theta) + (1 - \frac{r_+}{r})\theta \tag{44}$$

$$P_\theta = \frac{cr_+}{16\pi r^3}(1-\theta) + \frac{c}{16\pi r_+}\delta \tag{45}$$

By combining the above solutions with the help of the weighted superposition principle discussed in section 4.1, many interior solutions for a black hole with given gravitational mass M can be constructed. One can start out with any power in the mass-density $\rho = c_i/(8\pi r^{(2+i)})$ and get a valid solution which obeys the equation of state $\rho + P_r = 0$. For any mass-density ρ which has a power-expansion in the radial coordinate r, a solution obeying the above equation of state can then be constructed by the weighted superposition principle of section 4.

Not all of these solutions will be realized physically. At the current state of knowledge, i.e. lacking a detailed account of the physical and mathematical properties of the new solutions, it will be quite impossible to give precise rules for the selection of the physically interesting solutions. The holographic principle might provide a valuable guideline. According to the holographic principle, first formulated by t'Hooft in [33] and extended by Susskind [32], a three-dimensional gravitational system should be describable solely in terms of the properties of its two-dimensional boundary.

The following list of requirements should be viewed as a first attempt to formulate selection principles for the physically interesting solutions of classical general relativity along the lines of the holographic principle:

- weak holographic selection principle:

  All properties of the classical space-time which are measurable by an external observer, such as gravitational mass or charge, should be explainable solely in terms of the properties of the boundary surface of a self gravitating object.

Therefore the "stress-energy content" of the membrane situated at the boundary should be equal to the gravitating mass of the object.

- strong holographic selection principle

  A somewhat stronger requirement is the following: The sum of all matter-fields of a self-gravitating object, i.e. $\rho + P_r + 2P_\theta$, i.e. its active gravitational energy-density, should be zero everywhere, except at the boundary. Furthermore the trace of the stress-energy tensor of all fields, integrated over the full space-time, should be equal to the gravitating mass of the object.

For the above derived solutions the "weak holographic selection principle" requires that the tangential pressure be of the following form:

$$P_\theta = f(r)(1-\theta) + \frac{1}{16\pi r_+}\delta \tag{46}$$

This leads to the condition $c = 1$, which is satisfied for a wide class of solutions and thus is too weak to restrict the solutions to a manageable number. If we require the point mass at the origin to be zero, however, we get $n = 0$ from equation (42), which gives us the singular metric solution.

In order to apply the "strong holographic selection principle" the sum of all matter fields, i.e. the active gravitational energy-density, is evaluated:

$$\rho + P_r + 2P_\theta = 2P_\theta = \frac{cn}{8\pi r^2}(\frac{r_+}{r})^n(1-\theta) + \frac{c}{8\pi r_+}\delta \tag{47}$$

If this sum is to be zero everywhere except at the boundary, the condition $n = 0$ follows.[14]

Thus by invoking the strong and weak holographic selection principles formulated above, only one solution ($c = 1$, $n = 0$) remains. The "singular metric solution" of chapter 4.4 has been recovered.

Note that the "singular metric" solution can also be singled out by formulating constraints with respect to the 00-component of the Ricci-tensor:

$$R_{00} = -4\pi B(\rho + P_r + 2P_\theta)$$

The R_{00}-component of the Ricci-Tensor is identical zero throughout the *whole* space-time for the "singular metric" solution (the delta-distribution is cancelled by the zero in B). Thus demanding that R_{00} be zero everywhere leads to the condition $n = 0$, however doesn't fix the value of c.

In the more general case ($\rho \neq c/(8\pi r^2)$; $rho + P_r \neq 0$) the interior pressure is locally anisotropic. Generally, at any position $r \neq 0$ the radial pressure component will differ from the two tangential pressure components. In the case $P_\theta \neq 0$ the following difficulty arises: At $r = 0$ both P_r and P_θ can point in any direction, thus P_θ must be considered as radial pressure component. If P_r

[14] This remains true for arbitrary ρ, whenever ρ can be expressed as a power-expansion in r, because the basis functions r^n are linearly independent

and P_θ have different both non-zero values at $r = 0$, there are two mutually incompatible values for the pressure at this particular point. In a pure classical context this would lead to the rejection of all solutions except for those, where one of the pressure components is zero or both are equal at $r = 0$. However, the failure of classical concepts at a single space-time point should not be viewed as an unsurmountable problem. According to loop quantum gravity any region of space-time with a (closed) boundary surface smaller than roughly the Planck-surface should be viewed as devoid of interior structure and thus inaccessible for measurement. Geometric operators referring to a point are undefined in quantum gravity [29].

4.7 Compactar solutions with equation of state $\rho + P_r = 0$ and a power law in ρ

In this section I extend the holographic solution of section 4.5 to mass-densities which follow an arbitrary power law in the radial coordinate r. The only difference to the previous section is, that $r_h \neq r_+$.

With the mass density given by equation (40) B is derived by a simple integration:

$$B = \big(1 - \frac{2M_0}{r} - \frac{c}{1-n}(\frac{r_+}{r})^n\big)(1-\theta) + \big(1 - \frac{r_+}{r}\big)\theta \tag{48}$$

The metric must be continuous. Therefore we can determine the position of the boundary r_h as the position, where interior and exterior metrics are equal:

$$r_h = r_+\big(\frac{1-n}{c}(1 - \frac{2M_0}{r_+})\big)^{\frac{1}{1-n}} \tag{49}$$

Any exponent $n > 1$ will either require a negative value of c, i.e. a negative energy-density ρ or a point mass $M_0 > r_+$, in order to get a real valued solution.

Under the assumption that M_0 is universal and small and r_+ generally is large, M_0 must be negative and the coefficient $\frac{1-n}{c}$ must be positive and greater or equal to one, if the boundary r_h of the matter distribution shall lie outside of its gravitational radius r_+.

With the continuity equation (16) the tangential pressure can be calculated:

$$P_\theta = \frac{cn}{16\pi r^2}(\frac{r_+}{r})^n(1-\theta) + \frac{c}{16\pi r_h}(\frac{r_+}{r_h})^n\delta \tag{50}$$

It is possible to replace the term in front of the delta-distribution as follows:

$$P_\theta = \frac{cn}{16\pi r^2}(\frac{r_+}{r})^n(1-\theta) + \frac{c}{16\pi r_h}c^{\frac{2}{1-n}}(1-n)^{\frac{n+1}{n-1}}(1 - \frac{2M_0}{r_+})^{\frac{n+1}{n-1}}\delta \tag{51}$$

For large values of n the tangential pressure will tend to:

$$\lim_{n\to\infty} P_\theta = \frac{cn}{16\pi r^2}(\frac{r_+}{r})^n(1-\theta) + \frac{c(1-n)}{16\pi r_+}(1 + \frac{r_0}{r_+})\delta \tag{52}$$

with

$$r_0 = -2M_0$$

The "strong holographic selection principle" in combination with the equation of state, $\rho + P_r = 0$, requires the interior region to be free of tangential pressure. This leads to $n = 0$. If the membrane is to carry an energy content equal to the gravitating mass of the compactar, and if $r_0 \ll r_+$, $c = 1$ follows. So when the selection principle is applied to the compactar solutions, we recover the holographic solution.

5 Solutions with equation of state $P_r = a\rho$

In this section we would like to somewhat relax the constraint on the equation of state. The main consideration for choosing an equation of state linear in P_r and ρ is, that a linear relationship renders a great part of the problem solvable in terms of elementary functions.

It might appear to the reader that a linear equation of state is overly restrictive. On the other hand, for highly gravitating systems with highly relativistic particle momenta we should expect that the pressure becomes comparable to the mass-density. Furthermore any length-, mass- or time-scales related to the electro-weak or strong forces will not be able to exert any noticable influence on the physics of high gravitational fields. The relationship between mass-density and pressure(s) should only depend on dimensional considerations. The "natural" dimensional relation between mass-density and the pressure is a linear one: In units $c = 1$ both have the same dimension.

In this section I will show that - for given ρ - it is possible to derive any solution with an equation of state $P_r = a\rho$ from the special solution with a vanishing radial pressure. The full class of solutions with a linear equation of state therefore only depends on the mass-density ρ, on the integration constant M_0 (or rather r_0) and on the constant of proportionality between pressure and mass-density, a.

The metric coefficients of the solution with zero radial pressure ($a = 0$) will be denoted by A and B. The metric coefficients of the general solution ($a \neq 0$) will be denoted by $A^{(a)}$ and $B^{(a)}$, respectively. Likewise the tangential pressure for the general case is denoted by ${P_\theta}^{(a)}$ and the tangential pressure for the special case ($a = 0$) by P_θ.

A does not depend on the radial pressure. Thus the radial metric coefficient A is independent of the equation of state parameter a: $A^{(a)} = A$. We therefore drop the superscript in A. Only $B^{(a)}$ and ${P_\theta}^{(a)}$ need to be determined.

It is easy to derive $B^{(a)}$:

$$AB^{(a)} = e^{-\int_r^\infty 8\pi r A(\rho + P_r)dr} = e^{-(1+a)\int_r^\infty 8\pi r A \rho dr} = (AB)^{1+a}$$

$$B^{(a)} = A^a B^{1+a} \tag{53}$$

We find that $B^{(a)}$ can be expressed by simple powers of A and B, which is the metric for a radial pressure of zero. For $a = -1$ the known result $B^{(-1)} = 1/A$ is recovered.

The tangential pressure ${P_\theta}^{(a)}$ can be derived from the tangential pressure of the zero (radial)pressure case. The calculation is best done using the continuity equation. The result is:

$$ {P_\theta}^{(a)} = (1+a)P_\theta + a\left((\rho + \frac{r\rho'}{2}) + (1+a)2\pi r^2\rho^2 A\right) \tag{54} $$

Lets take a closer look at the case $a = -1$. Due to the factor $1 + a$ in (54) the above expression is highly simplified:

$$ {P_\theta}^{(-1)} = -(\rho + \frac{r\rho'}{2}) = -\frac{(r^2\rho)'}{2r} \tag{55} $$

For $a = -1$ the tangential pressure can be derived exclusively from the mass density ρ and it's first derivative.

By setting $\rho = \frac{c}{8\pi r^2}(\frac{r_+}{r})^n(1-\theta)$, the results found in sections 4.6 and 4.7 for $a = -1$ are recovered:

$$ {P_\theta}^{(-1)} = \frac{nc}{16\pi r^2}(\frac{r_+}{r})^n(1-\theta) + \frac{c}{16\pi r_+}\delta \tag{56} $$

However, equation (55) is more general than the result found in sections 4.6 and 4.7. The mass-density ρ isn't limited to a power-expansion in r. Therefore equation (55) immediately tells us, that - besides the trivial case $\rho = 0$ - only an inverse square law for the mass-density leads to a zero tangential pressure inside the entire source-region. We can also see, that whenever the mass-distribution is discontinuous, the tangential pressure will have a δ-distribution at the position of the discontinuity.

If $a \neq -1$ the field equations are non-linear and the weighted superposition principle of section 3 cannot be applied. It is not possible to construct a general solution by power-expanding ρ in the radial coordinate value r. Nevertheless the solutions with $\rho \propto r^m$ provide valuable hints with respect to the properties of more general solutions.

6 Solutions with $\rho \propto r^m$

In this section I describe a general procedure for the derivation of solutions with a mass-density following a power-law in r. The mass-density is expressed as:

$$ \rho = \frac{c}{8\pi r^2}\left(\frac{r_h}{r}\right)^n(1-\theta) \tag{57} $$

The radial metric coefficient for such a mass-density follows from integration of equation (9). It is independent of the equation of state:

$$ A(r) = \frac{1}{1 - \frac{r_h - r_0}{r} + \frac{c}{n-1}((\frac{r_h}{r})^n - \frac{r_h}{r})}(1-\theta) + \frac{1}{1 - \frac{r_h - r_0}{r}}\theta \tag{58} $$

The starting point of the integration has been chosen to be the position of the boundary, r_h. The first term with $(1-\theta)$ describes the interior metric and corresponds to an integration in the inward direction. The second exterior term (with θ) follows from integration in the outward direction. A positive integration constant r_0 has been assumed at the boundary:

$$r_0 = \frac{r_h}{A(r_h)}$$

If the solution is to possess an event horizon, $r_0 = 0$. Compactar solutions have $r_0 > 0$.

As long as the mass-density doesn't contain a δ-distribution the metric is continuous across the boundary. By comparing the exterior metric with the Schwarzschild metric we find:

$$r_h - r_0 = r_+ = 2M$$

The time coefficient of the metric can be determined from equation (8) as follows:

$$\ln AB(r) = (1+a)8\pi \int_{r_h}^{r} rA\rho \, dr + \ln AB(r_h) \tag{59}$$

Due to the vanishing mass-density in the exterior space-time, the exterior metric has $\ln AB(r) = \ln AB(r_h) = \mathit{const}$. Comparing the exterior metric to the Schwarzschild-metric gives $AB(r \geq r_h) = 1$. Therefore the integration constant $\ln AB(r_h)$ must be zero, if the metric is to be continuous across the boundary and the exterior space-time is to be described by the Schwarzschild metric.

For an arbitrary mass-density the integral in equation (59) for the interior metric cannot be expressed in terms of elementary functions. For certain integer-powers a closed form of the solution can be given. The most interesting cases are discussed in the following sections.

6.1 Solutions with $\rho = aP_r$ and a mass-density $\rho \propto 1/r^2$

Let us assume an interior mass-density of the following form:

$$\rho(r) = \frac{c}{8\pi r^2}(1-\theta) \tag{60}$$

The integration of equation (9), starting out from the boundary r_h with $r_h/A(r_h) = r_0$ yields:

$$\frac{r}{A} = \big(r_0 + (1-c)(r-r_h)\big)(1-\theta) + \big(r - (r_h - r_0)\big)\theta \tag{61}$$

From this A can be determined:

$$A(r) = \frac{r}{r_0}\left(1 - (1-c)\frac{r_h - r}{r_0}\right)^{-1}(1-\theta) + \left(1 - \frac{r_h - r_0}{r}\right)^{-1}\theta \tag{62}$$

The exterior part of the solution (the term involving θ) is set equal to the Schwarzschild-metric with:

$$r_+ = 2M = r_h - r_0 \tag{63}$$

r_+ is the gravitational radius and M the total gravitating mass.

The radial metric coefficient A is continuous across the boundary. A reaches its maximum value at the boundary:

$$A_{max} = A(r_h) = \frac{r_h}{r_0} = 1 + \frac{r_+}{r_0} = 1 + \frac{2M}{r_0}$$

Therefore the maximum value of A, at the position of the boundary r_h, scales linearly with the mass of the compactar, measured in Planck-units. A compactar/black hole of solar mass has a gravitational radius of roughly 3 km. Therefore we have: $A(r_h) \cong 3km/l_{Pl} \approx 10^{38}$.

The time coefficient of the metric B can be calculated by integrating equation (8). The integration is again performed from the boundary, r_h:

$$\ln A(r)B(r) - \ln A(r_h)B(r_h) = \int_{r_h}^{r} 8\pi r A(r)(\rho + P_r)dr$$

The case $1 + a = 0$ will not be discussed here. It has been discussed in detail in the previous sections. For $c \neq 1$ we get:

$$\frac{AB(r)}{AB(r_h)} = (1 - (1-c)\frac{r_h - r}{r_0})^{-\frac{(1+a)c}{c-1}}(1-\theta) + \theta \tag{64}$$

According to Birkhoff's theorem the exterior space-time ($\rho = 0$) should be described by the Schwarzschild metric. Therefore $AB(r_h)$ must be 1.

B is finally given by:

$$B(r) = \frac{r_0}{r}(1 - (1-c)\frac{r_h - r}{r_0})^{1-\frac{(1+a)c}{c-1}}(1-\theta) + (1 - \frac{r_h - r_0}{r})\theta \tag{65}$$

At the boundary, B is just the inverse of A. Therefore $B(r_h)$ can become very small for large compactars. For a compactar of stellar mass $B(r_h) \approx 10^{-38}$.

The tangential pressure is calculated via equation (14):

$$P_\theta = \frac{(1+a)c}{32\pi r^2}\left((1+ac)A(r) - 1\right)(1-\theta) - \frac{ac}{16\pi r_h}\delta \tag{66}$$

By expanding A we get the final result:

$$P_\theta = \frac{(1+a)c}{32\pi r^2}\left(-1 + (1+ac)\frac{r}{r_0}\left(1 - (1-c)\frac{r_h - r}{r_0}\right)^{-1}\right)(1-\theta) - \frac{ac}{16\pi r_h}\delta \tag{67}$$

For $c = 1$:

$$A(r) = \frac{r}{r_0}(1-\theta) + \frac{1}{1 - \frac{r_h - r_0}{r}}\theta \tag{68}$$

$$B(r) = \frac{r_0}{r} e^{-(1+a)\frac{r_h - r}{r_0}}(1-\theta) + (1 - \frac{r_h - r_0}{r})\theta \tag{69}$$

$$P_\theta = \frac{1+a}{32\pi r^2}\left(-1 + (1+a)\frac{r}{r_0}\right)(1-\theta) - \frac{a}{16\pi r_h}\delta \tag{70}$$

The pressures and mass-densities at the boundary r_h are given by:

$$\rho(r_h) = \frac{c}{8\pi {r_h}^2} \tag{71}$$

$$P_r(r_h) = a\rho(r_h) = \frac{ac}{8\pi {r_h}^2} \tag{72}$$

$$P_\theta(r_h) = \frac{(1+a)c}{4}\left(-1 + (1+ac)\frac{r_h}{r_0}\right)\rho(r_h) - \frac{ac}{16\pi r_h}\delta \tag{73}$$

Note that for $c \neq 1$ the possible values of r_h are somewhat restricted. The radial metric coefficient $A(r)$ and the time coefficient $B(r)$ should remain positive and real-valued throughout the whole physically meaningful interior region of the black hole. This leads to the following inequality:

$$1 - (1-c)\frac{r_h - r}{r_0} \geq 0 \tag{74}$$

The "physically meaningful interior region" shall be defined as $r > r_0$.[15] If the above inequality is to be satisfied by all interior r-values in the range between $r_0 \leq r \leq r_h$ we get the following inequality:

$$\frac{r_h}{r_0} \leq \frac{2-c}{1-c} \tag{75}$$

or

$$c \geq 1 - \frac{r_0}{r_h - r_0} = \frac{1 - 2\frac{r_0}{r_h}}{1 - \frac{r_0}{r_h}} \tag{76}$$

If the interior mass-density is given by an inverse square law in r, large compactars with $r_h \gg r_0$ are only possible if $c \to 1$. For any compactar of a given size r_h there is a minimum value of c, given by equation (76), which very rapidly approaches 1 for large r_h.

We can interpret this as tentative evidence, that large compactars should have an interior mass-density very close to the "natural" mass density $\rho = 1/(8\pi r^2)$.

[15] According to the discussion in [22] is appears more appropriate to choose $r > r_0/2$. A slightly different choice of the "physically meaningful interior region" affects the results only quantitatively. The reader can easily make the necessary adjustments.

Compactar solutions with a mass-density $\rho \propto 1/r^2$ are of particular interest. The holostar solution with $\rho = 1/(8\pi r^2)$ will be discussed in detail in [23]. The holostar solution, in contrast to the more general solutions, can be generalized in a natural manner to encompass charged matter. The charged holostar solution is discussed in [22]. The interesting thermodynamic properties are discussed in [24]. Some interesting properties of the more general inverse square law solutions will be discussed in a forthcoming paper.

6.2 Solutions with $P_r = a\rho$ and $\rho \propto 1/r^4$

Such solutions correspond to $n = 2$ in equation (57). The radial metric coefficient has already been calculated at the beginning of chapter 6.

For $\rho \propto 1/r^4$ solutions for all values of a and r_0 can be expressed in terms of elementary functions. The integration of the time-coefficient of the metric gives:

$$\ln AB = \frac{1+a}{2}\left(\frac{\gamma}{\beta}\ln \delta \frac{(\beta+\gamma)\frac{r_h}{r} - 2}{(\beta-\gamma)\frac{r_h}{r} + 2} - \ln \frac{r_h}{r_0}(1 - \gamma\frac{r_h}{r} + c\frac{r_h^2}{r^2})\right) \tag{77}$$

with

$$\gamma = 1 + c - \frac{r_0}{r_h} \tag{78}$$

$$\beta^2 = \gamma^2 - 4c \tag{79}$$

$$\delta = \frac{\beta + (2-\gamma)}{\beta - (2-\gamma)} \tag{80}$$

B is calculated by exponentiating equation (77).

The tangential pressure P_θ can be calculated from the metric by a simple differentiation. It suffices to calculate P_θ for $a = 0$ and to derive the tangential pressure for $a \neq 0$ by the procedure described in section 5. The calculation is straightforward, and can be done either by hand or with a symbolic math program.

6.3 Other solutions

It is not possible to discuss, or even present, the full solution space to the general spherically symmetric problem in a single paper. For an arbitrary mass-density ρ and an arbitrary equation of state $P_r(\rho) = 0$, equations (9 , 8) have to be integrated numerically. For some particular cases solutions in terms of elementary functions are possible. For $n = -1$, i.e. $\rho \propto 1/r$, there exists a full solution for all values of c and r_0 for a linear equation of state. For $n = 3$ and $n = -2$ black hole solutions (with $r_0 = 0$) can be expressed in a closed form. For $n = -3$ there is a solution for $r_0 = 0$ and $c = 1$.

7 Discussion

A new class of solutions to the field equations of general relativity has been presented. An unexpected property of the new solutions is the appearance of a localizable two-dimensional membrane at the generally non-continuous boundary of the matter distribution. The membrane has a zero mass-density, but considerable surface tension/pressure, hinting at a "new" state of matter in high gravitational fields. However, if and how the new solutions can contribute to a better understanding of the physical phenomena of the Small and the Large remains a question, which can only be answered in full by future research.

A field of research which presents itself immediately is the the generalization of the solutions discussed in this paper to the rotating and / or charged case. The charged holostar solution is discussed in [22]. The search for a solution describing a compact rotating body will be a challenging topic of future research.

The holographic solution appears to have some potential to be a good approximation to our physical world. Its geometric properties are discussed in detail in [23]. In a parallel paper [24] it is shown, that the entropy/area law for black holes and the Hawking temperature follow as a direct consequence of the holographic interior metric ($g_{rr} = r/r_0$), combined with microscopic statistical thermodynamics.

Even if the holographic solution turns out to be incompatible with the properties of the real world, it might still be of some interest in another respect: There are indications that pressure effects cannot be neglected in a realistic description of gravitational phenomena, neither in the gravitational collapse of a star, nor on a cosmological scale. For neutron stars anisotropic models are already on the table. The simple mathematics of the holographic solution will enable us to study some of the consequences of pressure-induced effects and therefore might provide valuable insights with respect to more realistic space-times including (anisotropic) pressure.

The new solutions presented in this paper are exact mathematical solutions of the field equations of general relativity. Any solution that cannot be ruled out by convincing physical arguments has a reasonable chance to be actually realized by nature. Due to the high energies involved in self gravitational phenomena, the usual scientific approach to choose among different mathematical possibilities by well controlled experiments will not be feasible, probably not even possible. It appears that ultimatively we will have to decide by observation alone, which of the physically acceptable solutions - if any - have been selected by nature. Most likely the study of collision processes of compact self gravitating objects, possibly the study of accretion processes, will eventually provide unambiguous answers. Observation of such effects will require sophisticated space- and ground-based equipment. As long as conclusive observational evidence is lacking, theoretical research will fill have to fill in the gaps.

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A The geodesic equations of motion for a spherically symmetric system

If one studies the properties of a solution to the field equations, it is often helpful to analyze the geodesic motion of particles within the space-time. Note however, that in space-times with significant pressure pure geodesic motion is not possible.

The purpose of this chapter is to express the geodesic equations of motion for a general spherically symmetric system in a "more geometric" form that will prove useful for future discussions.

In any spherically symmetric space-time the equations of geodesic motion can be expressed in terms of the metric-coefficients A and B (see for example [12, p. 135]):

$$\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{A}\left(\frac{l^2}{r^2} + \epsilon\right) - \frac{F^2}{AB} = 0 \tag{81}$$

$$\frac{d\varphi}{d\tau} r^2 = l = const \tag{82}$$

$$\frac{dt}{d\tau} B = F = const \tag{83}$$

ϵ is zero for a particle of zero rest-mass and 1 for a massive particle. l is the constant angular momentum (per particle mass) and F is a positive constant related to the total energy of the motion of a massive particle.

It is quite helpful to express the motion of the particles as fraction of the local velocity of light at a particular radial position. The local velocity of light, expressed in terms of the (t, r, θ, φ) coordinate values, can be read off from the metric. The local velocity of light in the radial direction (c_r) and in the tangential direction ($c_\perp$) is given by:

$$c_r = \sqrt{\frac{B}{A}} \tag{84}$$

$$c_\perp = \sqrt{B} \tag{85}$$

The argument r in the above quantities has been omitted.

If we denote by $\beta_r(r)$ the local radial velocity of the particle, i.e. expressed as ratio to the local speed of light in the radial direction, and by $\beta_\perp(r)$ the respective local tangential velocity, we find:

$$\beta_r(r) = \frac{dr}{dt}/c_r = \frac{dr}{dt}/\sqrt{\frac{B}{A}} \tag{86}$$

and

$$\beta_\perp(r) = \frac{rd\varphi}{dt}/c_\perp = \frac{rd\varphi}{dt}/\sqrt{B} = \frac{l}{F}\frac{\sqrt{B}}{r} \tag{87}$$

With this notation the radial equation of motion (81) can be written as a sum of a kinetic and potential energy term:

$$\beta_r^2(r) + V_{eff}(r) = 1 \tag{88}$$

with

$$V_{eff}(r) = \frac{B(r)}{F^2}(\frac{l^2}{r^2} + \epsilon) = \beta_\perp^2(r) + \epsilon\frac{B(r)}{F^2} \tag{89}$$

From equation (89) one can see, that the effective potential can be expressed as the sum of two terms. The first term, $\beta_\perp^2(r)$, is the square of the local tangential velocity. The second term is only relevant for particles of non-zero rest-mass. It involves F^2 and $B(r)$.

At any turning point of the motion, r_i, the radial velocity is zero, and therefore according to equation (88), $V_{eff}(r_i) = 1$. This allows us to express the constants of the motion, l and F, by two parameters whose geometric interpretation is more evident: the radial position of the turning point, r_i, and the local tangential velocity at the turning point, $\beta_\perp^2(r_i) = \beta_i^2$:

$$\frac{l^2}{F^2} = \frac{r_i^2}{B(r_i)}\beta_i^2 \tag{90}$$

and

$$\frac{\epsilon}{F^2} = \frac{1-\beta_i^2}{B(r_i)} = \frac{1}{\gamma_i^2 B(r_i)} \tag{91}$$

For particles with zero rest mass (photons) only equation (90) is relevant. Equation (91) is trivially fulfilled. For photons the local tangential velocity at the turning point of the motion must be equal to the local speed of light in the tangential direction, i.e. $\beta_i^2 = 1$. For particles with non zero rest-mass ($m_0 \neq 0$) the factor $1 - \beta_i^2 = 1/\gamma_i^2$ is nothing else than the squared ratio of the particle's rest mass to its local total relativistic energy at the turning point of the motion, E_i. Therefore equation (91) can be expressed as follows:

$$F^2 = B(r_i)\gamma_i^2 = B(r_i)\left(\frac{E_i}{m_0}\right)^2 \tag{92}$$

In a pressure-free, stationary space-time the local energy of a particle, measured by observers at different coordinate positions is related by the gravitational Doppler-shift factor, $\sqrt{g_{00}(r_1)/g_{00}(r_2)}$. For an asymptotically flat space time $g_{00}(r=\infty) = B(\infty) = 1$. Under these circumstances the constant of the motion F can be identified with the local total energy of the particle at its turning point of the motion, divided by its rest-mass, as measured by an observer at rest at spatial infinity.

The constant of the motion l can be expressed as follows for a particle of non-zero rest mass,

$$l^2 = r_i^2 \frac{\beta_i^2}{1-\beta_i^2} = r_i^2 \beta_i^2 \gamma_i^2 = r_i^2 \left(\frac{p_i}{m_0} \right)^2 \tag{93}$$

where p_i is the relativistic momentum of the particle at the turning point of the motion. The geometric interpretation of l is quite evident: The constant angular momentum J of the particle is given by $J = lm_0 = p_i r_i$, i.e. is proportional to the local linear momentum p times the radial coordinate distance from the center r, both taken at the turning point of the motion, r_i.

For further reference it is useful to express the local radial and tangential velocities of particles undergoing geodesic motion in the following form, involving only the "geometric" constants of the motion r_i, β_i and the metric coefficient B:

$$\beta_\perp^2(r) = \frac{B(r)}{B(r_i)} \frac{r_i^2}{r^2} \beta_i^2 \tag{94}$$

$$\beta_r^2(r) + V_{eff}(r) = 1 \tag{95}$$

$$V_{eff}(r) = \frac{B(r)}{B(r_i)} \left(1 - \beta_i^2 (1 - \frac{r_i^2}{r^2}) \right) \tag{96}$$